

CHARACTERIZATION OF COMPACT MONOTONICALLY (ω)-MONOLITHIC SPACES USING SYSTEM OF RETRACTIONS

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ABSTRACT. We prove that a compact space is monotonically Sokolov if and only if it is monotonically ω -monolithic. This gives answers to several questions of R. Rojas-Hernández and V. V. Tkachuk.

1. INTRODUCTION

Spaces with a rich family of retractions often occur both in topology and functional analysis. In Banach space theory, using systems of retractions we can obtain a system of projections and consequently find a Markushevich basis; see e.g. [1]. In topology, Gul'ko used families of retractions in [4] to prove that a compact space K is Corson whenever $\mathcal{C}_p(K)$ has the Lindelöf Σ -property. The method of Gul'ko's proof was further studied and precised in [9].

One of the possible concepts of a family of retractions was recently introduced in [7]. Spaces having such a system were called monotonically Sokolov and using them, an answer to Problem 3.8 from [8] was given.

In this note we give a positive answer to Question 6.3 from [7], i.e. we prove that a compact space is monotonically Sokolov if and only if it is monotonically ω -monolithic.

Theorem 1. *Let K be a compact space. Then the following conditions are equivalent:*

- (i) *K is monotonically monolithic*
- (ii) *K is monotonically ω -monolithic*
- (iii) *K is monotonically Sokolov*

As a consequence answers to Questions 6.4 and 6.5 from [7] easily follow.

Corollary 2. *If K is a compact Collins-Roscoe space, then it is monotonically Sokolov.*

Corollary 3. *There exists a compact space K such that it is monotonically Sokolov but not Gul'ko.*

2. PRELIMINARIES

We denote by ω the set of all natural numbers (including 0). If X is a set then $\exp(X) = \{Y; Y \subset X\}$.

All topological spaces are assumed to be Hausdorff. Let T be a topological space. The closure of a set A we denote by \overline{A} . We denote the topology of T by $\tau(T)$ and $\tau(x, T) =$

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$\{U \in \tau(T); x \in U\}$ for any $x \in T$. A family \mathcal{N} of subsets of T is an *external network* of A in T if for any $a \in A$ and $U \in \tau(a, T)$ there exists $N \in \mathcal{N}$ such that $a \in N \subset U$.

Given an infinite cardinal κ say that a space T is *monotonically κ -monolithic* if, to any set $A \subset T$ with $|A| \leq \kappa$, we can assign an external network $\mathcal{O}(A)$ to the set \overline{A} in such a way that the following conditions are satisfied:

- (i) $|\mathcal{O}(A)| \leq |A| + \omega$;
- (ii) if $A \subset B \subset T$ and $|B| \leq \kappa$ then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
- (iii) if $\lambda \leq \kappa$ is a cardinal and we have a family $\{A_\alpha; \alpha < \lambda\} \subset [X]^{\leq \kappa}$ such that $\alpha < \beta < \lambda$ implies $A_\alpha \subset A_\beta$ then $\mathcal{O}(\bigcup_{\alpha < \lambda} A_\alpha) = \bigcup_{\alpha < \lambda} \mathcal{O}(A_\alpha)$.

The space T is *monotonically monolithic* if it is monotonically κ -monolithic for any infinite cardinal κ .

Topological space T is a *Collins-Roscoe space* if for each $x \in T$, one can assign a countable family $\mathcal{O}(x)$ of subsets of T such that, for any $A \subset T$, $\bigcup\{\mathcal{O}(x); x \in A\}$ is an external network for \overline{A} .

Let Γ be a set. We put $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma \in \Gamma : x(\gamma) \neq 0\}| \leq \omega\}$. A compact space K is *Corson compact* if there is a homeomorphic embedding of K into $\Sigma(\Gamma)$ for some set Γ .

Definition. Let X, Y be sets, $\mathcal{O} \subset \exp(X)$ closed under countable increasing unions, $\mathcal{N} \subset \exp(Y)$ and $f : \mathcal{O} \rightarrow \mathcal{N}$. We say that f is ω -monotone if

- (i) $f(A)$ is countable for every countable $A \in \mathcal{O}$;
- (ii) if $A \subset B$ and $A, B \in \mathcal{O}$ then $f(A) \subset f(B)$;
- (iii) if $\{A_n; n \in \omega\} \subset \mathcal{O}$ and $A_n \subset A_{n+1}$ for every $n \in \omega$ then $f(\bigcup_{n \in \omega} A_n) = \bigcup_{n \in \omega} f(A_n)$.

Definition. A space T is *monotonically Sokolov* if we can assign to any countable family \mathcal{F} of closed subsets of T a continuous retraction $r_{\mathcal{F}} : T \rightarrow T$ and a countable external network $\mathcal{N}(\mathcal{F})$ for $r_{\mathcal{F}}(T)$ in T such that $r_{\mathcal{F}}(F) \subset F$ for every $F \in \mathcal{F}$ and the assignment \mathcal{N} is ω -monotone.

3. PROOFS OF THE MAIN RESULTS

The following proposition is the key tool to prove Theorem 1. The idea of the proof is moreover in following the lines of the proof Lemma 2.4 (a) of [6]. In order to obtain the ω -monotonicity, we use a fixed “Skolem function” (see e.g. [2, Section 2]) to construct the elementary submodels from [6, Lemma 2.4].

Proposition 4. *Let K be a Corson compact space. Then, to any countable family \mathcal{F} of closed subsets of K we can assign a countable set $M(\mathcal{F}) \subset K$ and a retraction $r_{\mathcal{F}}$ such that*

- (i) $r_{\mathcal{F}}(F) \subset F$ for every $F \in \mathcal{F}$,
- (ii) $r_{\mathcal{F}}(K) = \overline{M(\mathcal{F})}$ and
- (iii) the assignment $\mathcal{F} \mapsto M(\mathcal{F})$ is ω -monotone.

Proof. In the proof we denote by \mathcal{B} the set of all the rational open intervals in \mathbb{R} . Without loss of generality we may assume that $K \subset \Sigma(\Gamma)$ for some set Γ . If $\gamma_1, \dots, \gamma_n \in \Gamma$ and $U_1, \dots, U_n \in \mathcal{B}$, we put

$$[\gamma_1, \gamma_2, \dots, \gamma_n; U_1, U_2, \dots, U_n] = \{x \in \mathbb{R}^\Gamma : x(\gamma_i) \in U_i \text{ for any } i = 1, 2, \dots, n\}.$$

For $S \subset K$ we denote by $\text{supp}(S)$ the set of all $\gamma \in \Gamma$ such that $s(\gamma) \neq 0$ for some $s \in S$. Note that $\text{supp}(S)$ is countable whenever $S \subset K$ is countable. For $x \in K$ and $A \subset \Gamma$ we

denote by $x \upharpoonright_A$ the point in $\Sigma(\Gamma)$ defined by $x \upharpoonright_A (\gamma) = x(\gamma)$ for $\gamma \in A$ and $x \upharpoonright_A (\gamma) = 0$ for $\gamma \in \Gamma \setminus A$.

If F is a non-empty closed subset of K , then we pick a point $x_F \in F$. For any $k \in \mathbb{N}$, $\gamma_1, \dots, \gamma_k \in \Gamma$ and $U_1, \dots, U_k \in \mathcal{B}$ we pick, if it exists, $x(F, \gamma_1, \dots, \gamma_k; U_1, \dots, U_k) \in F \cap [\gamma_1, \dots, \gamma_k; U_1, \dots, U_k]$.

Take a countable family \mathcal{F} of closed subsets of K . We will recursively construct $M(\mathcal{F})$. Let $M_0(\mathcal{F}) = \{x_F; F \in \mathcal{F} \cup \{K\}\}$. Assume that $n \in \omega$ and we have countable sets $M_0(\mathcal{F}), \dots, M_n(\mathcal{F})$. Let

$$M_{n+1}(\mathcal{F}) = M_n(\mathcal{F}) \cup \bigcup_{F \in \mathcal{F}} \left\{ x(F, \gamma_1, \dots, \gamma_k; U_1, \dots, U_k); \gamma_1, \dots, \gamma_k \in \text{supp}(M_n(\mathcal{F})), \right. \\ \left. U_1, \dots, U_k \in \mathcal{B}, k \in \mathbb{N} \right\}.$$

Notice that $M_{n+1}(\mathcal{F}) \subset K$ is countable since the set $M_n(\mathcal{F})$ is countable. We will prove that $M(\mathcal{F}) = \bigcup \{M_n(\mathcal{F}); n \in \omega\}$ and $r_{\mathcal{F}}(x) = x \upharpoonright_{\text{supp}(M(\mathcal{F}))}$, $x \in K$ are as promised.

Claim 1. $r_{\mathcal{F}}(F) \subset \overline{F \cap M(\mathcal{F})}$ for every $F \in \mathcal{F} \cup \{K\}$

Proof. Take an arbitrary $F \in \mathcal{F} \cup \{K\}$, $x \in F$ and $W \in \tau(r_{\mathcal{F}}(x), [0, 1]^\Gamma)$. There are $j \in \mathbb{N}$, $\gamma_1, \dots, \gamma_j \in \Gamma$ and $U_1, \dots, U_j \in \mathcal{B}$ such that $V = [\gamma_1, \dots, \gamma_j; U_1, \dots, U_j] \subset W$ and $r_{\mathcal{F}}(x) \in V$. It suffices to find some $y \in F \cap M(\mathcal{F}) \cap V$.

If $\{\gamma_1, \dots, \gamma_j\} \cap \text{supp}(M(\mathcal{F})) = \emptyset$ then we put $y = x_F$. It is immediate that $y \in F \cap M(\mathcal{F})$. Moreover, since $y \in M(\mathcal{F})$, $y(\gamma) = 0$ for $\gamma \in \Gamma \setminus \text{supp}(M(\mathcal{F})) \supset \{\gamma_1, \dots, \gamma_j\}$; hence, $y(\gamma_i) = 0 = r_{\mathcal{F}}(x)(\gamma_i)$ for every $i \in \{1, \dots, j\}$. Thus, $y \in V$.

Otherwise, find $k \in \mathbb{N}$ and i_1, \dots, i_k such that

$$\{\gamma_{i_1}, \dots, \gamma_{i_k}\} = \{\gamma_1, \dots, \gamma_j\} \cap \text{supp}(M(\mathcal{F})).$$

Now it is enough to put $y = x(F, \gamma_{i_1}, \dots, \gamma_{i_k}, U_{i_1}, \dots, U_{i_k})$ and observe that then $y \in F \cap M(\mathcal{F}) \cap V$. \square

From the claim above it immediately follows that $r_{\mathcal{F}} : K \rightarrow K$ is a continuous retraction, $r_{\mathcal{F}}(K) \subset \overline{M(\mathcal{F})}$ and $r_{\mathcal{F}}(F) \subset F$ for every $F \in \mathcal{F}$. Notice, that whenever $x \in M(\mathcal{F})$, $\text{supp}(x) \subset \text{supp}(M(\mathcal{F}))$ and hence $r_{\mathcal{F}}(x) = x$. Consequently, $M(\mathcal{F}) \subset r_{\mathcal{F}}(K)$ and $r_{\mathcal{F}}(K) = \overline{M(\mathcal{F})}$.

Claim 2. The assignment $\mathcal{F} \mapsto M(\mathcal{F})$ is ω -monotone.

Proof. It takes a straightforward induction to see that the set $M(\mathcal{F})$ is countable for any countable family \mathcal{F} of closed subsets of K and the assignments $\mathcal{F} \mapsto M_n(\mathcal{F})$ are ω -monotone for every $n \in \omega$. Now it is easy to observe, e. g. by [7, Proposition 4.3], that the assignment $\mathcal{F} \mapsto M(\mathcal{F}) = \bigcup \{M_n(\mathcal{F}); n \in \omega\}$ is ω -monotone. \square

\square

Proof of Theorem 1. It is immediate that (i) \Rightarrow (ii). Suppose that K is monotonically ω -monolithic. It follows from [3, Corollary 2.2] that K must be Corson; hence, e.g. by [5, Lemma 1.6], it has a countable tightness. By [10, Theorem 2.10], any monotonically ω -monolithic space of countable tightness is monotonically monolithic; hence, we proved (ii) \Rightarrow (i). The implication (iii) \Rightarrow (ii) is proved in [7, Proposition 4.4].

Finally, suppose that K is monotonically ω -monolithic. By [3, Corollary 2.2], K is Corson. Now we can apply Proposition 4 to convince ourselves that for any countable family \mathcal{F} of closed subsets of K we can choose a countable set $M(\mathcal{F}) \subset K$ and a retraction $r_{\mathcal{F}}$ such that $r_{\mathcal{F}}(K) = \overline{M(\mathcal{F})}$, $r_{\mathcal{F}}(F) \subset F$ for any $F \in \mathcal{F}$ and the assignment M is ω -monotone. Since K is monotonically ω -monolithic, to each countable set $S \subset K$ we can assign a countable family $\mathcal{O}(S) \subset \exp(K)$ which is an external network of \overline{S} in such a way that \mathcal{O} is ω -monotone. Let $\mathcal{N}(\mathcal{F}) = \mathcal{O}(M(\mathcal{F}))$. Then $\mathcal{N}(\mathcal{F})$ is a countable external network of $r_{\mathcal{F}}(K) = \overline{M(\mathcal{F})}$ in K and the assignment \mathcal{N} is ω -monotone because it is a composition of ω -monotone mappings. Hence, K is monotonically Sokolov and (ii) \Rightarrow (iii) follows. \square

Proof of Corollary 2. This is an easy consequence of Theorem 1 because every Collins-Roscoe space is monotonically monolithic; see e.g. [3, Lemma 3.1]. \square

Proof of Corollary 3. By [10, Example 3.12], there exists a compact Collins-Roscoe space which is not Gul'ko. By Corollary 2, every compact Collins-Roscoe is monotonically Sokolov. \square

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REFERENCES

- [1] Marek Cúth. Simultaneous projectional skeletons. *J. Math. Anal. Appl.*, 411(1):19–29, 2014.
- [2] Marek Cúth and Ondřej F. K. Kalenda. Rich families and elementary submodels. accepted in Cent. Eur. J. Math. (2014), preprint available at <http://arxiv.org/abs/1308.1818>.
- [3] Gary Gruenhage. Monotonically monolithic spaces, Corson compacts, and D -spaces. *Topology Appl.*, 159(6):1559–1564, 2012.
- [4] S. P. Gul'ko. The structure of spaces of continuous functions and their hereditary paracompactness. *Uspekhi Mat. Nauk*, 34(6(210)):33–40, 1979.
- [5] Ondřej F. K. Kalenda. Valdivia compact spaces in topology and Banach space theory. *Extracta Math.*, 15(1):1–85, 2000.
- [6] Wiesław Kubiś and Henryk Michalewski. Small Valdivia compact spaces. *Topology Appl.*, 153(14):2560–2573, 2006.
- [7] R. Rojas-Hernández and V. V. Tkachuk. A monotone version of the Sokolov property and monotone retractability in function spaces. *J. Math. Anal. Appl.*, 412(1):125–137, 2014.
- [8] Vladimir V. Tkachuk. A nice class extracted from C_p -theory. *Comment. Math. Univ. Carolin.*, 46(3):503–513, 2005.
- [9] Vladimir V. Tkachuk. Condensing function spaces into Σ -products of real lines. *Houston J. Math.*, 33(1):209–228 (electronic), 2007.
- [10] Vladimir V. Tkachuk. Lifting the Collins-Roscoe property by condensations. *Topology Proc.*, 42:1–15, 2013.

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